Dynamics of a many-particle Landau-Zener model: inverse sweep

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We consider dynamics of a slowly time-dependent Dicke model, which represents a many-body generalization of the Landau-Zener model. In particular, the model describes narrow Feshbach resonance passage in an ultracold gas of Fermi atoms. Adiabaticity is destroyed when a parameter crosses a critical value, even at very slow sweeping rates of a parameter. The dynamics crucially depends on direction of the sweep. We apply our recent analysis [A.P. Itin, P. Torma, arXiv:0901.4778v1] to the "inverse" sweep through the resonance, corresponding (in a context of Feshbach resonance passage) to dissociation of molecules. On a level of the mean-field approximation, the dynamics is equivalent to a molecular condensate formation from Bose atoms within a two-mode model. Mapping the system to a Painlevé equation allows us to calculate deviation from adiabaticity at very slow sweeps analytically.

Destruction of adiabaticity is conceptually important issue in both quantum and classical mechanics. Nonadiabatic dynamics of many quantum single-particle systems can be effectively described in terms of the exactly solvable Landau-Zener model [1] (LZM), where a probability of transition from an initially occupied instantaneous ground state level to an excited one is exponentially small in a sweeping rate parameter. Numerous studies have been devoted to such type of models [2].

Many-particle quantum systems are intriguing objects that can exhibit nonlinear behavior and in some sense link quantum and classical worlds together. Interest to such systems has grown enormously due to progress in ultracold quantum gases experiments [3]. Destruction of adiabaticity in semiclassical models of many-particle systems under slow driving has been intensively investigated recently [4, 5, 6, 7, 8, 9] using various methods. It has been found that exponential LZM-type behavior for transition probabilities is often replaced with power-laws of sweeping rates. Theoretical methods used for derivation of these power-laws often involve a number of approximations those accuracy is difficult to control.

We considered recently [10] a time-dependent Dicke model (see Eq. 2 below) using concepts of classical adiabatic invariants. Technically, a method which naturally links classical mechanics and quantum dynamics of many-particle systems is a so-called Truncated Wigner Approximation (TWA) [7, 11, 12]. Briefly, within this approximation one considers an ensemble of classical trajectories whose initial conditions are distributed in phase space according to a Wigner transform of an initial quantum state. One therefore is left with the problem of nonadiabaticity of individual classical solutions. In classical mechanics, it is known that adiabaticity is destroyed when a phase point crosses or comes close to a separatrix of an unperturbed system [15]. There is a method for calculating changes in approximate adiabatic invariants: separatrix crossing theory [13, 14, 15]. The method has a drawback: usually it is not applicable in case a separatrix crossing happens close to or at the time of bifurcation, i.e. in case of very small initial action of a classical trajectory. In [10], we managed to overcome this drawback by mapping the system close to a bifurcation to a Painlevé equation, those asymptotics are known. A formula for change in the action we derived there allowed to describe mean values of number of bosons created at the resonance passage, and its distributions. In this Brief Report we consider the "inverse" sweep through the resonance in the same model. We also discuss other physical realizations of the semiclassical model, in particular association of Bose atoms in diatomic molecules within a simple two-model model [5].

Let us consider the time-dependent Dicke model written in the following form [7, 16]:

$$\hat{H} = -\Delta(t)b^\dagger b + \Delta(t)S^z + \frac{g}{\sqrt{N}}(b^\dagger S^- + bS^+), \quad (1)$$

where $\frac{g}{\sqrt{N}}$ is the coupling strength, $S^\pm = S_x \pm iS_y$ are spin operators, $b^\dagger$ and b are creation and destruction operators of a bosonic mode, $\Delta(t) = \pm\epsilon t$ is detuning, ϵ is the sweeping rate of the bosonic mode energy. $\Delta(t) = +\epsilon t$ for a "forward" sweep through resonance, and $-\epsilon t$ for a "backward", or "inverse", sweep. The spin value S is macroscopically large $S = N/2 \gg 1$. With $N = 2$, one recovers the standard Landau-Zener model. We set $g = 1$ for convenience in this paper.

For a "forward" sweep through the resonance considered in [10], we started in the distant past with some small initial number of bosons $N_b(t)|_{-\infty} \equiv \langle b^\dagger b \rangle(t)|_{-\infty} = N_- \equiv \mathbf{n}_- N$, and calculated the final number of bosons $N_b(t)|_{+\infty} \equiv \mathbf{n_f} N$ by averaging over an ensemble of classical trajectories. For a backward sweep, only numerical results were presented in Altland et al [7], consistent with the earlier work of [5]. Our method enables to explain them analytically with high accuracy. The classical limit of (1), obtained using the $c - number$ formalism [7],[9], is

$$H = -\gamma n - 2n\sqrt{1-n}\cos\phi, \quad \gamma = \pm 2\epsilon t, \quad (2)$$

where n corresponds to the number of bosons, and ϕ is the canonically conjugated phase. For a forward sweep, $\gamma = 2\epsilon t$, while for a backward sweep $\gamma = -2\epsilon t$. Note that by means of a trivial change of variables [17], the
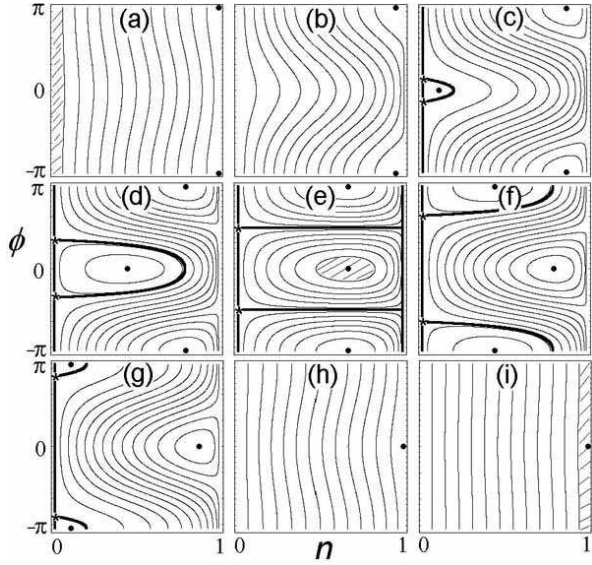
FIG. 1: Phase portraits of the Hamiltonian (2). From (a) to (f): $\gamma = -20, -4, -1.8, -0.9, 0., 0.9, 1.8, 20, 100$ correspondingly. Saddle points are denoted by asterisks, the bold trajectory is the separatrix. Shaded areas illustrate definitions of the classical actions (see main text).
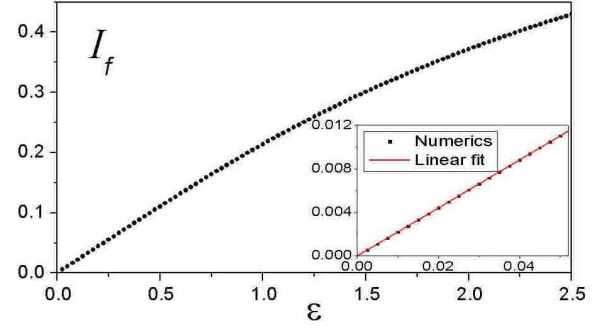

FIG. 2: Change in the classical action for the backward sweep with zero initial action (i.e., $\Delta I = I_f - I_- = I_f$). Inset shows change in the action for small values of ϵ and a linear fit which gives $I_f/\epsilon = 0.220631$. Theoretical prediction (7) is $\frac{\ln 2}{\pi} \approx 0.220636$.

Hamiltonian becomes the same as analyzed in [9] within the same context, and can be derived from semiclassical Bloch equations of [6]. A mean-field model of association of bosonic condensates into a molecular condensate considered in [5] is also equivalent to (2), with $n = 1$ corresponding to "all-atom" mode (i.e., on a mean-field level association of bosonic atoms is analogous to dissociation of molecular condensate of Fermi atoms [18]).

Let us recall the classical phase space of the Hamiltonian (2) at fixed values of the parameter γ, as explained in [9]. If $\gamma < -2$, there is only one stable elliptic point on the phase portrait (at $\phi = \pi$ and n not far from 1, see Fig.1a,b). At $\gamma = -2$, a bifurcation takes place, and at $|\gamma| < 2$ phase portraits look as shown in Figs.1c,d,e,f,g. There are two saddle points at $n = 0, \cos\phi = -\gamma/2$, and a newborn elliptic point at $\phi = 0$. The trajectory connecting these two saddles separates rotations and oscillating motions and it is called the *separatrix* of the frozen system [9]. Finally, at large positive γ, again there is only one elliptic stationary point at $\phi = 0$, and n close to 1. The classical action is defined as in [7, 9] and is shown graphically in Figs 1a,e,i: shaded areas, divided by 2π. At $\gamma = -\infty$, the action coincides with n: $I = n$. At $\gamma = +\infty$, we have $I = 1 - n$. For a forward sweep, starting at $t = -\infty$ with certain small $n = n_-$, in the adiabatic limit we get $n_+ = 1 - n_-$ at $t = +\infty$. Analogously, for a backward sweep, starting from a $n_- \approx 1$, in the adiabatic limit we get $n_+ = 1 - n_-$ at $t = +\infty$. Destruction of adiabaticity in both cases is mainly caused by motion near the saddle points located at $n = 0$, i.e. no matter in what direction a sweep happens, region of the loss of adiabaticity is the same. For a forward sweep,

a classical phase point with $I_- = 0$ (i.e., located initially at $n = 0$) always remains there. Therefore, when approximating an initial quantum state by an ensemble of classical trajectories, one needs to carefully take into account quantum fluctuations. For a ground state as an initial state, initial actions of the classical ensemble $\sim 1/N$. For a backward sweep, the situation is different. A phase point with $I_- = 0$ is not stationary and therefore in the semiclassical limit, one may merely study its classical dynamics. For small sweeping rates, dynamics remains adiabatic until the phase point approaches a saddle point at $n = 0$. Then, following [10], close to a saddle point the system can be mapped to a Painleve equation. To this end, one introduces new variables $P = 2\sqrt{n}\cos\phi/2$, $Y = 2\sqrt{n}\sin\phi/2$. After the transformations and approximations described in [10] one obtains the Hamiltonian $H = -s\frac{Y^2}{2} + \frac{P^2}{2} + \frac{Y^4}{2}$, where s is a new time variable and P, Y are canonically conjugated variables. One can see that the effective Hamiltonian leads to the Painlevé-2 equation [19, 20]

$$\frac{d^2Y}{ds^2} = sY - 2Y^3. \qquad (3)$$

Asymptotics of the Painlevé-2 equation were investigated by Its and Kapaev [20]. At $s \to -\infty$ the asymptotic solution to (3) is [20, 21]

$$Y(s) = \alpha(-s)^{-\frac{1}{4}}\sin\left(\frac{2}{3}(-s)^{3/2} + \frac{3}{4}\alpha^2\ln(-s) + \varphi\right),$$

and in the limit $s \to +\infty$ it is

$$Y(s) = \pm\sqrt{\frac{s}{2}} \pm \rho(2s)^{-\frac{1}{4}}\cos\left(\frac{2\sqrt{2}}{3}s^{3/2} - \frac{3}{2}\rho^2\ln(s) + \theta\right),$$

where (α, φ) and (ρ, θ) are the "action-angle" variables characterizing the solutions in the limits $s \to \pm\infty$. These variables are related by the following relations [20, 21]:

$$\rho^2 = \frac{1}{\pi}\ln\frac{1 + |p|^2}{2|\text{Im}(p)|}$$

$$\theta = -\frac{\pi}{4} + \frac{7}{2}\rho^2 \ln 2 - \arg\Gamma(i\rho^2) - \arg(1+p^2), \quad (4)$$

$$p = \left(\exp(\pi\alpha^2)\right)^{1/2} \times$$
$$\times \exp\left(i\frac{3}{2}\alpha^2 \ln 2 - \frac{\pi}{4}i - i\arg\Gamma\left(i\frac{\alpha^2}{2}\right) - i\varphi\right),$$

where $\arg(\cdot)$ denotes the argument of a complex number, and $\Gamma(x)$ is the gamma function. As $s \to \pm\infty$, the adiabatic invariant I_p of the system (3) approaches the quantities I_p^- or I_p^+ which are defined (to the lowest order terms) as

$$I_p^- = \frac{\alpha^2}{2}, \quad I_p^+ = \frac{\rho^2}{2}. \quad (5)$$

Relations (4),(5) allow to calculate the change in the action both for a forward [10] and backward sweep.

It turns out that the backward sweep through the resonance has very different behaviour from the forward one. Most importantly, in the limit $I_- \to 0$ change in the action becomes phase-independent. That is, at large $I_- \gg \epsilon$ predictions of the Painlevé mapping correspond to the separatrix crossing theory calculations. However, as I_- decreases, the change in the action approaches the phase independent value, which is linearly proportional to ϵ.

Indeed, for a classical trajectory with $I_- = 0$ from (4),(5) it follows [21]

$$I_p^- = \frac{1}{2\pi} \ln(1 + |p|^2) \quad (6)$$

For the backward sweep, I_p^- corresponds to I_f, while I_p^+ corresponds to I_-. For the initial action $I_- = 0$ we have $|p| = 0$. Taking into account all the transformations leading to the Painlevé equation, we obtain:

$$\Delta I = I_f = \frac{\ln 2}{\pi}\epsilon \quad (7)$$

This is the main result of the present paper. Numerical calculations (see figure 2) reproduce the coefficient $\frac{\ln 2}{\pi}$ with 5-digit accuracy. In Ref. [5], a linear power-law for deviation from adiabaticity was also found numerically, however the accuracy of the analytical solution reported was lower than in our case.

To conclude, we found that the mapping of the many-particle Landau-Zener model to the Painlevé equation [10] enables to describe not only "forward" sweep through the resonance, but also the reverse driving process. The latter process arises, for example, when the model is used to describe formation of diatomic molecules from Bose atoms, or dissociation of diatomic molecules on Fermi atoms. Our results are relevant for accurately describing the Feshbach resonance passage in ultracold Fermi and Bose gases. Interferometric experiments with ultracold quantum gases [24] is another possible area of applications. Finally, we believe the method will have important applications in the field of *dynamics of quantum phase transitions* (QPT) [22, 23, 25].

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[17] $\tilde{n} = w = 1-2n, \tilde{\phi} = \pi-\phi, \tilde{H} = \sqrt{2}H, \tilde{t} = \sqrt{2}t, \delta = \gamma/\sqrt{2}$. Omitting tildes, $H = \delta(w-1) + (1-w)\sqrt{1+w}\cos\phi$.

[18] Ref. [5] considered system $\{i\dot{\alpha} = \frac{\Omega}{\sqrt{2}}\alpha^*\beta, \quad i\dot{\beta} = \frac{\Omega}{\sqrt{2}}\alpha^2 + \delta\beta\}$, with $|\alpha|^2 = 1$ corresponding to the "all-atom" mode. Denoting $n = |\alpha|^2$, $\phi = 2\arg(\alpha) - \arg(\beta)$, after simple rescalings we get the Hamiltonian (2).

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